2018

A note on L2-boundary integrals of the Bergman kernel

Trong Thuc Phung

University of Science-HCMC VNU, Vietnam, University of Wollongong, ttp754@uowmail.edu.au

A note on L2-boundary integrals of the Bergman kernel

Abstract
In this paper, we obtain some estimates on the L2-boundary norm of the Bergman kernel for pseudoconvex domains admitting a plurisubharmonic defining function.

Keywords
integrals, l2-boundary, bergman, note, kernel

Disciplines
Engineering | Science and Technology Studies

Publication Details
Trong Thuc, P. (2018). A note on L2-boundary integrals of the Bergman kernel. International Journal of Mathematics, 29 (14), 1871001-1-1871001-16.

A NOTE ON L^2-BOUNDARY INTEGRALS OF THE BERGMAN KERNEL

PHUNG TRONG THUC

ABSTRACT. For any bounded convex domain Ω with C^2 boundary in $\mathbb{C}^n$, we show that there exist positive constants C_1 and C_2 such that

$$C_1\sqrt{\frac{K(w,w)}{\delta(w)}} \leqslant \|K(\cdot,w)\|_{L^2(\partial\Omega)} \leqslant C_2\sqrt{\frac{K(w,w)}{\delta(w)}},$$

for any $w \in \Omega$. Here K is the Bergman kernel of Ω, and δ is the distance-to-boundary function.

1. Introduction

Throughout this note, Ω will be a bounded pseudoconvex domain in $\mathbb{C}^n$, and $\delta : \Omega \to \mathbb{R}^+$ is the distance function to the boundary, $\delta(z) := \inf\{\|z-w\| : w \in \partial\Omega\}$. The Bergman kernel $K(z,w)$ of Ω is the reproducing kernel for the space of square-integrable holomorphic functions. That is, $K : \Omega \times \Omega \to \mathbb{C}$ is the function such that

$$f(z) = \int_\Omega K(z,w) f(w)\, dV(w), \forall f \in A^2(\Omega).$$

Here $A^2(\Omega)$ denotes the space of square-integrable holomorphic functions on Ω, and dV is Lebesgue measure.

It was suggested by Catlin ([11]) for studying the boundary behavior of the Bergman kernel, by starting with simple domains, such as ellipsoids. We are interested in obtaining estimates on the L^2-boundary norm of the Bergman kernel for convex domains. When Ω is a smoothly convex domain, the function $K(\cdot,w)$ is smooth up to the boundary, for any fixed $w \in \Omega$, due to the work of Boas and Straube, see [10]. Thus the boundary integral $\int_{\partial\Omega} |K(z,w)|^2\, d\sigma(z)$ is well-defined. Here $d\sigma$ is the standard surface measure. Even if we drop the smoothness of $\partial\Omega$, we can still conclude that $K(\cdot,w) \in L^2(\partial\Omega)$ in the trace sense. This follows from the fact that any bounded convex domain has a Lipschitz boundary ([19]), and the Bergman projection is bounded in the Sobolev space $W^{1/2}(\Omega)$, see [27]. To avoid technical reasons, we will restrict ourselves to the case $\partial\Omega \in C^2$. The purpose of this paper is to estimate the norm $\|K(\cdot,w)\|_{L^2(\partial\Omega)}$ as w varies in Ω.

The main result is stated as follows.

Theorem 1.1. *Let Ω be a bounded convex domain with C^2 boundary in $\mathbb{C}^n$. Then*

$$C_1\sqrt{\frac{K(w,w)}{\delta(w)}} \leqslant \|K(\cdot,w)\|_{L^2(\partial\Omega)} \leqslant C_2\sqrt{\frac{K(w,w)}{\delta(w)}}, \tag{1.1}$$

for any $w \in \Omega$. Here C_1 is a positive constant depending on Ω, and $C_2 = \sqrt{4en+1}$.

2010 *Mathematics Subject Classification.* Primary 32A25; Secondary 32W05.

Key words and phrases. Bergman kernel; Pluricomplex Green function; Weighted Bergman projections; L^2-boundary integrals.

The author was supported by PhD scholarship in ARC grant DE140101366.

Our method is elementary, basically being based on a weighted version of Bergman projections by Berndtsson and Charpentier ([4]), and relations between the pluricomplex Green function and the Bergman kernel. Our approach is motivated by the work of Chen and Fu ([15]) on the comparison of the Bergman and Szegö kernels. In fact, by the definition of the Szegö kernel, from Theorem 1.1 we obtain that for any bounded convex domain Ω in $\mathbb{C}^n$,

$$\frac{S(z,z)}{K(z,z)} \geqslant \frac{\delta(z)}{4en+1}, \forall z \in \Omega.$$

Here S is the Szegö kernel of Ω. We note that in [15], the authors obtained estimates on S/K for a much larger class of domains called δ-regular, which includes pseudoconvex domains of finite type and pseudoconvex domains having plurisubharmonic defining functions at boundary points. Our argument here can be extended to domains admitting a plurisubharmonic defining function, see Remark 4.3. It seems to us that our approach and the stated result (Theorem 1.1) have not been noticed in the literature.

In Section 2, we provide a weighted version of Bergman projections. We recall some properties of the pluricomplex Green function in Section 3. The proof of Theorem 1.1 is given in the last section.

Notation. We will use the notation $X \lesssim Y$ (resp. $X \gtrsim Y$) to denote the estimate $|X| \leqslant CY$ (resp. $X \geqslant C|Y|$), for some positive constant C. We use $X \approx Y$ for the fact $X \lesssim Y \lesssim X$.

Acknowledgements. The author thanks his supervisors: Jiakun Liu and Tran Vu Khanh for helpful suggestions and encouragement. The author would like to thank Professor Bo-Yong Chen for his comments.

2. Weighted Bergman projections

Let ψ be a Lebesgue measurable function on Ω. By $L^2(\Omega, e^{\psi})$ we denote the Hilbert space of measurable functions associated with the norm

$$\|f\|_{L^2(\Omega,e^{\psi})} := \sqrt{\int_\Omega |f|^2 e^{\psi} dV}.$$

Let P be the Bergman projection associated to Ω. It can be represented as

$$P(f)(z) := \int_\Omega K(z,w) f(w) dV(w), \forall f \in L^2(\Omega). \tag{2.1}$$

The formula (2.1) allows us to extend the domain of definition of P. Let f be a Lebesgue measurable function on Ω, we say that $P(f)$ is well-defined on Ω if for almost every $z \in \Omega$, we have $K(z,\cdot) f(\cdot) \in L^1(\Omega)$. For example, when Ω is a smoothly bounded pseudoconvex domain of finite type, $P(f)$ is well-defined for any $f \in L^p(\Omega)$, with $p \geqslant 1$ (see, e.g. [9, 3]).

The main purpose of this section is to establish the following result.

Proposition 2.1. *Let Ω be a bounded pseudoconvex domain. Let $0 < r < 1$ and let ψ be a locally bounded, plurisubharmonic function on Ω such that $ri\partial\overline{\partial}\psi \geqslant i\partial\psi \wedge \overline{\partial}\psi$ as currents. Then for any Lebesgue measurable function f so that $P(f)$ is well-defined, the following inequality holds*

$$\int_\Omega |P(f)(z)|^2 e^{\psi(z)} dV(z) \leqslant \frac{1}{1-r} \int_\Omega |f(z)|^2 e^{\psi(z)} dV(z). \tag{2.2}$$

Remark 2.2. The inequality (2.2) was already stated in [4] with an implicit constant in the right-hand side. It turns out that the constant $1/(1-r)$ here allows us to establish the estimates in Theorem 1.1.

We also recall that the condition $r\imath\partial\overline{\partial}\psi \geqslant \imath\partial\psi\wedge\overline{\partial}\psi$ is equivalent to the statement that $-e^{-\psi/r}$ is plurisubharmonic on Ω.

Proof. It suffices to consider the case that the right hand side of (2.2) is finite. We will employ a similar approach as in [4]. The idea is to shift the standard L^2 space to weighted one by Kohn's formula and by being more careful with the use of weighted Bergman projections.

We first assume that $\psi \in C^2\left(\overline{\Omega}\right)$ and $f \in L^2(\Omega)$. By Kohn's formula (see e.g. [24, 16])

$$P(f) = e^{-\psi}P_\psi\left(e^\psi f\right) - \overline{\partial}^\star N\left(\overline{\partial}\left(e^{-\psi}P_\psi\left(e^\psi f\right)\right)\right), \tag{2.3}$$

where P_ψ denotes the Bergman projection in $L^2\left(\Omega, e^{-\psi}\right) = L^2(\Omega)$. Set $g := e^{-\psi}P_\psi\left(e^\psi f\right)$ and $u := \overline{\partial}^\star N\left(\overline{\partial}\left(e^{-\psi}P_\psi\left(e^\psi f\right)\right)\right)$. Since $P_\psi\left(e^\psi f\right) \in A^2(\Omega)$, we have $\int_\Omega u\overline{g}e^\psi = 0$. Thus

$$\int\limits_\Omega |P(f)|^2 e^\psi = \int\limits_\Omega |g-u|^2 e^\psi = \int\limits_\Omega |g|^2 e^\psi + \int\limits_\Omega |u|^2 e^\psi. \tag{2.4}$$

For the first term of (2.4), we get

$$\int\limits_\Omega |g|^2 e^\psi = \int\limits_\Omega \left|P_\psi\left(e^\psi f\right)\right|^2 e^{-\psi} \leqslant \int\limits_\Omega \left|e^\psi f\right|^2 e^{-\psi} = \int\limits_\Omega |f|^2 e^\psi. \tag{2.5}$$

Note that $u = \overline{\partial}^\star N\left(-g\wedge\overline{\partial}\psi\right)$, and ue^ψ belongs to the orthogonal complement of $\ker\overline{\partial}$ in $L^2\left(\Omega, e^{-\psi}\right)$. Thus we obtain

$$\int\limits_\Omega \left|ue^\psi\right|^2 e^{-\psi} \leqslant \int\limits_\Omega \left|\overline{\partial}\left(ue^\psi\right)\right|^2_{\imath\partial\overline{\partial}\psi} e^{-\psi}. \tag{2.6}$$

It continues

$$\begin{aligned}\int\limits_\Omega |u|^2 e^\psi &\leqslant \int\limits_\Omega \left|\overline{\partial}u + \overline{\partial}\psi\wedge u\right|^2_{\imath\partial\overline{\partial}\psi} e^\psi \\ &= \int\limits_\Omega \left|-g\wedge\overline{\partial}\psi + \overline{\partial}\psi\wedge u\right|^2_{\imath\partial\overline{\partial}\psi} e^\psi \\ &\leqslant r\int\limits_\Omega \left(|u|^2 + |g|^2\right) e^\psi,\end{aligned} \tag{2.7}$$

here the last inequality follows by $\int_\Omega u\overline{g}e^\psi = 0$ and $\left|\overline{\partial}\psi\right|^2_{\imath\partial\overline{\partial}\psi} \leqslant r$. Therefore

$$\int\limits_\Omega |u|^2 e^\psi \leqslant \frac{r}{1-r}\int\limits_\Omega |g|^2 e^\psi. \tag{2.8}$$

From (2.4), (2.5) and (2.8), we get the estimate (2.2).

When $f \in L^2(\Omega)$, but ψ is just a locally bounded, plurisubharmonic on Ω. Consider a sequence of pseudoconvex domains $\{\Omega_j\}$ such that $\overline{\Omega}_j \Subset \Omega_{j+1}$ and $\Omega = \bigcup_{j=1}^\infty \Omega_j$. For $\varepsilon > 0$, denote the convolution $\psi_\varepsilon := \psi \star \eta_\varepsilon$ the standard regularization. For each j, we can choose ε_j such that $0 < \varepsilon_j < \operatorname{dist}(\Omega_j, \partial\Omega)$ and

$$\int\limits_{\Omega_j} |f|^2 \left(e^{\psi_{\varepsilon_j}} - e^\psi\right) \leqslant \frac{1}{j}. \tag{2.9}$$

This is due to the monotone convergence theorem and the fact $f \in L^2(\Omega) \cap L^2(\Omega, e^{\psi})$. Since $ri\partial\overline{\partial}\psi_{\varepsilon_j} \geqslant i\partial\psi_{\varepsilon_j} \wedge \overline{\partial}\psi_{\varepsilon_j}$, by applying the previous argument, we get that

$$\int_{\Omega_j} \left|P_{\Omega_j}(f)\right|^2 e^{\psi_{\varepsilon_j}} \leqslant \frac{1}{1-r}\int_{\Omega_j} |f|^2 e^{\psi_{\varepsilon_j}}.$$

By $\psi_{\varepsilon_j} \geqslant \psi$ on Ω_j and (2.9), it continues

$$\int_{\Omega_j} \left|P_{\Omega_j}(f)\right|^2 e^{\psi} \leqslant \frac{1}{1-r}\int_{\Omega_j} |f|^2 e^{\psi_{\varepsilon_j}} \leqslant \frac{1}{1-r}\left(\frac{1}{j} + \int_{\Omega} |f|^2 e^{\psi}\right). \tag{2.10}$$

We conclude that for each fixed $k \in \mathbb{Z}^+$, the sequence $\left\{P_{\Omega_j}(f)\right\}_{j=k}^{\infty}$ is uniformly bounded in $L^2(\Omega_k)$. By Cantor's diagonal argument, we can assume, by passing to a subsequence, that $P_{\Omega_j}(f)$ converges weakly to a function v in $L^2(\Omega, \mathrm{loc})$. It is clear that $P_{\Omega_j}(f)\, e^{\psi/2}$ also converges weakly to $ve^{\psi/2}$ in $L^2(\Omega, \mathrm{loc})$. Thus for each $K \Subset \Omega$, by (2.10)

$$\int_K |v|^2 e^{\psi} \leqslant \liminf_{j\to\infty} \int_K \left|P_{\Omega_j}(f)\right|^2 e^{\psi} \leqslant \frac{1}{1-r}\int_{\Omega} |f|^2 e^{\psi}.$$

It follows that

$$\int_{\Omega} |v|^2 e^{\psi} \leqslant \frac{1}{1-r}\int_{\Omega} |f|^2 e^{\psi}.$$

We now prove that $v = P_{\Omega}(f)$. First, since $\overline{\partial}\left(P_{\Omega_j}(f)\right) = 0$ and $P_{\Omega_j}(f) \to v$ weakly, v is holomorphic in Ω. It remains to show that

$$\int_{\Omega} |f-v|^2 \leqslant \int_{\Omega} |f-h|^2, \forall h \in A^2(\Omega). \tag{2.11}$$

To see this, fix any $K \Subset \Omega$, we have

$$\begin{aligned}
\int_K |f-v|^2 &\leqslant \liminf_{j\to\infty} \int_K \left|f - P_{\Omega_j}(f)\right|^2 \\
&\leqslant \liminf_{j\to\infty} \int_{\Omega_j} \left|f - P_{\Omega_j}(f)\right|^2 \\
&\leqslant \liminf_{j\to\infty} \int_{\Omega_j} |f-h|^2 \\
&\leqslant \int_{\Omega} |f-h|^2.
\end{aligned}$$

So (2.11) follows.

Finally, we consider the case when we only require that $\int_{\Omega} |f|^2 e^{\psi}$ is finite and $P(f)$ is well-defined. Set $f_k := \chi_{\Omega_k} f$, where the sequence $\{\Omega_k\}$ is the same as above, and χ_{Ω_k} is the indicator function of Ω_k. We have $f_k \in L^2(\Omega) \cap L^2(\Omega, e^{\psi})$ and

$$\int_{\Omega} |f_k - f|^2 e^{\psi} = \int_{\Omega\setminus\Omega_k} |f|^2 e^{\psi} \to 0 \text{ as } k \to \infty.$$

By the previous estimate,

$$\int_\Omega |P(f_k)|^2 e^\psi \leqslant \frac{1}{1-r} \int_\Omega |f_k|^2 e^\psi .$$

It follows that $\{P(f_k)\}$ is a Cauchy sequence in $L^2(\Omega, e^\psi)$ and so converges to a function v in $L^2(\Omega, e^\psi)$. Thus

$$\int_\Omega |v|^2 e^\psi \leqslant \frac{1}{1-r} \int_\Omega |f|^2 e^\psi .$$

Now, we only need to prove that $v = P(f)$. Since we can choose a subsequence of the $\{P(f_k)\}$ that converges pointwise to v, it suffices to show that $P(f_k)$ converges pointwise to $P(f)$. For each $z \in \Omega$,

$$\begin{aligned} |P(f_k)(z) - P(f)(z)| &= \left| \int_\Omega K(z,w) f_k(w)\, dw - \int_\Omega K(z,w) f(w)\, dw \right| \\ &= \left| \int_{\Omega \setminus \Omega_k} K(z,w) f(w)\, dw \right| . \end{aligned}$$

The last integral goes to zero as $k \to \infty$ since $K(z,\cdot) f(\cdot) \in L^1(\Omega)$.

□

Remark 2.3. The constant $1/(1-r)$ is not sharp for any $r \in (0,1)$. More specifically, it is not sharp for $r \approx 0$. To see this, by a result of Błocki ([5, p. 89]), for any function $v \perp \ker \overline{\partial}$ in $L^2(\Omega, e^{-\psi})$, we have

$$\int_\Omega |v|^2 e^{-\psi} \leqslant \frac{4r}{(1-r)^2} \int_\Omega \left|\overline{\partial} v\right|^2_{i\partial\overline{\partial}\psi} e^{-\psi}. \tag{2.12}$$

Now, apply (2.12) with $v := \overline{\partial}^\star N\left(-g \wedge \overline{\partial}\psi\right) e^\psi$, then we can replace the inequality (2.6) by

$$\int_\Omega \left|u e^\psi\right|^2 e^{-\psi} \leqslant \frac{4r}{(1-r)^2} \int_\Omega \left|\overline{\partial}\left(u e^\psi\right)\right|^2_{i\partial\overline{\partial}\psi} e^{-\psi}.$$

Continue the argument there, we get that

$$\int_\Omega |P(f)|^2 e^\psi \leqslant \frac{1}{1 - \frac{4r^2}{(1-r)^2}} \int_\Omega |f|^2 e^\psi ,$$

provided $0 < r < \frac{1}{3}$. Note that the constant $1/\left(1 - 4r^2/(1-r)^2\right)$ is sharper than $1/(1-r)$.

Nevertheless, if we call $C(r)$ the sharp constant for the estimate (2.2), that is, given $0 < r < 1$, $C(r)$ is the least constant such that for any pseudoconvex domain Ω and $ri\partial\overline{\partial}\psi \geqslant i\partial\psi \wedge \overline{\partial}\psi$, we have

$$\int_\Omega |P(f)|^2 e^\psi \leqslant C(r) \int_\Omega |f|^2 e^\psi ,$$

then we can show that

$$\lim_{r\to 1} \frac{C(r)}{\frac{1}{1-r}} = 1. \tag{2.13}$$

Therefore, the constant $1/(1-r)$ is sharp in the use of $r \to 1$. To see (2.13), choose $\Omega = \mathbb{D}$ the unit disc in $\mathbb{C}$, $f = (-\log|z|)^r$ and $\psi = -r\log(-\log|z|)$. We can easily check that

$$\frac{\int_\Omega |P(f)|^2 e^{\psi}}{\int_\Omega |f|^2 e^{\psi}} = \frac{\pi r}{\sin(\pi r)}.$$

Thus

$$\frac{\pi r}{\sin(\pi r)} \leqslant C(r) \leqslant \frac{1}{1-r},$$

and (2.13) follows.

Remark 2.4. By a duality argument, under the same hypothesis as in Proposition 2.1, we also have

$$\int_\Omega |P(f)(z)|^2 e^{-\psi(z)} dV(z) \leqslant \frac{1}{1-r}\int_\Omega |f(z)|^2 e^{-\psi(z)} dV(z).$$

Proposition 2.1, together with an idea of Chen ([14]), gives the following result.

Corollary 2.5. *Let Ω be a bounded pseudoconvex domain with a positive Diederich-Fornaess index η. Then for any $0 < t < \eta$ and $1 \leqslant q < 4n/(2n-t)$, the Bergman projection P associated to Ω is bounded from $L^2(\Omega, \delta^{-t})$ to $L^q(\Omega)$.*

Proof. Since $4n/(2n-t) > 2$, it suffices to assume that $q > 2$. Recall that the Diederich-Fornaess index of Ω is defined by

$$\eta(\Omega) := \sup\left\{\alpha \in [0,1] : \exists h \in PSH(\Omega) \text{ and } C > 0 \text{ such that } \frac{1}{C}\delta^\alpha < -h < C\delta^\alpha\right\}.$$

Thus, we can choose $t' \in (t, \eta)$ and $h \in PSH(\Omega)$ such that $\frac{1}{C}\delta^{t'} < -h < C\delta^{t'}$, for some positive constant C. Set $\psi := -(t/t')\log(-h)$, then $\psi \in L^\infty_{\text{loc}}(\Omega)$ and

$$\frac{t}{t'} i\partial\bar\partial\psi \geqslant i\partial\psi \wedge \bar\partial\psi.$$

For any measurable function f such that $\int_\Omega |f|^2 \delta^{-t}$ is finite (which also implies $f \in L^2(\Omega)$), by applying Proposition 2.1, we get that

$$\int_\Omega |P(f)|^2 \delta^{-t} \lesssim \int_\Omega |P(f)|^2 e^{\psi} \lesssim \int_\Omega |f|^2 e^{\psi} \lesssim \int_\Omega |f|^2 \delta^{-t}. \tag{2.14}$$

Thus, for any $\varepsilon > 0$,

$$\begin{aligned}\int_{\delta\leqslant\varepsilon} |P(f)|^2 &\leqslant \varepsilon^t \int_{\delta\leqslant\varepsilon} |P(f)|^2 \delta^{-t} \leqslant \varepsilon^t \int_\Omega |P(f)|^2 \delta^{-t} \\ &\lesssim \varepsilon^t \int_\Omega |f|^2 \delta^{-t}.\end{aligned}$$

Moreover, by the mean value inequality,

$$\begin{aligned}|P(f)(z)|^2 &\lesssim \delta^{-2n}(z) \int_{B(z,\delta(z))} |P(f)|^2 \\ &\lesssim \delta^{-2n}(z) \int_{\delta\leqslant 2\delta(z)} |P(f)|^2\end{aligned}$$

$$\lesssim \delta^{-2n+t}(z)\int_\Omega |f|^2\,\delta^{-t}.$$

It follows that for each $k \in \mathbb{Z}^+$,

$$\begin{aligned}\int_{2^{-k-1}<\delta\leqslant 2^{-k}} |P(f)|^q &\lesssim 2^{-(k+1)(q-2)\left(-n+\frac{t}{2}\right)}\left(\int_\Omega |f|^2\,\delta^{-t}\right)^{\frac{q}{2}-1}\int_{\delta\leqslant 2^{-k}}|P(f)|^2\\ &\lesssim 2^{-k\left(t-(q-2)\left(n-\frac{t}{2}\right)\right)}\left(\int_\Omega |f|^2\,\delta^{-t}\right)^{\frac{q}{2}}.\end{aligned}$$

Similarly,

$$\begin{aligned}\int_{\delta>1/2} |P(f)|^q &\lesssim \left(\frac{1}{2}\right)^{(q-2)\left(-n+\frac{t}{2}\right)}\left(\int_\Omega |f|^2\,\delta^{-t}\right)^{\frac{q}{2}-1}\int_\Omega |P(f)|^2\\ &\lesssim \left(\int_\Omega |f|^2\,\delta^{-t}\right)^{\frac{q}{2}-1}\int_\Omega |f|^2\\ &\lesssim \left(\int_\Omega |f|^2\,\delta^{-t}\right)^{\frac{q}{2}}.\end{aligned}$$

Therefore

$$\begin{aligned}\int_\Omega |P(f)|^q &\leqslant \int_{\delta>1/2}|P(f)|^q + \sum_{k=1}^{\infty}\int_{2^{-k-1}<\delta\leqslant 2^{-k}}|P(f)|^q\\ &\lesssim \left(1+\sum_{k=1}^{\infty}2^{-k\left(t-(q-2)\left(n-\frac{t}{2}\right)\right)}\right)\left(\int_\Omega |f|^2\,\delta^{-t}\right)^{\frac{q}{2}}\\ &\lesssim \left(\int_\Omega |f|^2\,\delta^{-t}\right)^{\frac{q}{2}},\end{aligned}$$

where the last inequality follows by the hypothesis $q < 4n/(2n-t)$. □

Note that we do not impose any regularity assumption on the boundary of Ω in Corollary 2.5. In the case when $\partial\Omega$ is Lipschitz, it is known that $\eta(\Omega)$ is positive, see [20]. Moreover, using the Lipschitz property, we can conclude that $\delta^{-\alpha} \in L^1(\Omega)$ for any $\alpha < 1$ (see e.g. [19]). Thus, using Hölder's inequality, we get that if $p \in (2,\infty)$ and $tp/(p-2) < 1$ then

$$\int_\Omega |f|^2\,\delta^{-t} \leqslant \left(\int_\Omega |f|^p\right)^{\frac{2}{p}}\left(\int_\Omega \delta^{\frac{-tp}{p-2}}\right)^{\frac{p-2}{p}} \lesssim \left(\int_\Omega |f|^p\right)^{\frac{2}{p}}.$$

Combining this with Corollary 2.5, we conclude that for a given $q \in [2, 4n/(2n-\eta))$, if there exists t such that

$$\eta > t > 2n - \frac{4n}{q} \text{ and } 1 - \frac{2}{p} > t,$$

then the Bergman projection P maps from $L^p(\Omega)$ to $L^q(\Omega)$ continuously. This requirement on t is equivalent to $p > 2q/(q + 2n(2-q))$.

We therefore arrive at the following result:

Corollary 2.6. *Let Ω be a bounded pseudoconvex domain with Lipschitz boundary. Let η be the Diederich-Fornaess index of Ω. Then for any $q \in [2, 4n/(2n-\eta))$, the Bergman projection associated to Ω is bounded from $L^p(\Omega)$ to $L^q(\Omega)$, provided that $p > 2q/(q + 2n(2-q))$.*

Remark 2.7. Corollary 2.6 says that for domains with Lipschitz boundary, one can always gain the regularity of the output space to an exponent bigger than 2 (i.e. $L^q, q > 2$), given that the regularity of the input is high enough. This is not the case for non-Lipschitz domains. For instance, consider the Hartogs triangle domain $\Omega_\gamma := \left\{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^\gamma < |z_2| < 1\right\}$, with $\gamma > 0$ and $\gamma \notin \mathbb{Q}$. This is a non-Lipschitz pseudoconvex domain (see e.g. [30]). It is known that for any $p \in (1, \infty)$ and $q > 2$, the Bergman projection associated to Ω_γ cannot be bounded from $L^p(\Omega_\gamma)$ to $L^q(\Omega_\gamma)$, see [18, p. 2681].

3. The pluricomplex Green function

We now recall some well-known results of the pluricomplex Green function. Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$. The pluricomplex Green function with a pole $w \in \Omega$ is defined by

$$G(\cdot, w) := \sup\left\{u(\cdot) : u \in PSH^-(\Omega), \limsup_{z \to w}(u(z) - \log|z-w|) < \infty\right\}.$$

Here $PSH^-(\Omega)$ denotes the set of all negative plurisubharmonic functions on Ω. The following results are used in the sequel.

Proposition 3.1 (Herbort [21], Błocki [8])**.** *Let Ω be a bounded pseudoconvex domain and let t be any positive number. Then*

(1) For any $f \in A^2(\Omega)$ and any $w \in \Omega$,

$$\int_{\{G(\cdot,w)<-t\}} |f(z)|^2\, dz \geqslant e^{-2nt}\frac{|f(w)|^2}{K(w,w)}. \tag{3.1}$$

(2) For any $w \in \Omega$,

$$K(w,w) \geqslant e^{-2nt} K_{\{G(\cdot,w)<-t\}}(w,w), \tag{3.2}$$

where $K_{\{G(\cdot,w)<-t\}}$ denotes the Bergman kernel of $\{G(\cdot,w) < -t\}$.

Proposition 3.2 (Chen [15], Błocki [6])**.** *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$. Assume that there exists a plurisubharmonic function φ on Ω such that for any $z \in \Omega$,*

$$C_1\delta^a(z) \leqslant -\varphi(z) \leqslant C_2\delta^b(z),$$

where C_1, C_2, a and b are positive constants. Then there exist positive constants C and δ_0 such that

$$\{G(\cdot,w) < -1\} \subset \left\{\frac{1}{C}\delta^{a/b}(w)\,|\log\delta(w)|^{-1/b} \leqslant \delta(\cdot) \leqslant C\delta^{b/a}(w)\,|\log\delta(w)|^{n/a}\right\}, \tag{3.3}$$

for any $w \in \Omega$ and $\delta(w) < \delta_0$.

If Ω is a convex domain then

$$\{G(\cdot,w) < -t\} \subset \left\{\frac{e^t-1}{e^t+1}\delta(w) \leqslant \delta(\cdot) \leqslant \frac{e^t+1}{e^t-1}\delta(w)\right\}, \tag{3.4}$$

for any $w \in \Omega$ and any $t > 0$.

For several applications of the pluricomplex Green function, we refer readers to [15, 6, 13, 14, 7]. The following result can be obtained by using these interesting properties.

Proposition 3.3. *Let $\Omega \subset \mathbb{C}^n$ be a strongly pseudoconvex domain with smooth boundary and let $\alpha \in \mathbb{R}$. If the Bergman-Toeplitz operator T_α, defined by*

$$f \to T_\alpha (f)(z) := \int_\Omega K(z,w) f(w) \delta^\alpha (w) dV(w),$$

is bounded from $L^p(\Omega)$ to $L^q(\Omega)$, with $1 < p \leqslant q < \infty$ then

$$\alpha \geqslant (n+1)\left(\frac{1}{p} - \frac{1}{q}\right).$$

Remark 3.4. This result has been obtained in [1] by using several estimates of Kobayashi balls and θ-Carleson measures in Bergman spaces. The proof given below is a direct consequence of Proposition 3.1 and Proposition 3.2. Note that the converse statement is also true, i.e. if $\alpha \geqslant (n+1)((1/p) - (1/q))$ then T_α maps from $L^p(\Omega)$ to $L^q(\Omega)$ continuously, see [29, 22].

Proof. Without loss of generality we may assume that $\alpha \geqslant 0$. First, it is well-known that for any strongly pseudoconvex domain Ω with smooth boundary, there is a positive constant $C(\Omega)$ such that

$$K(z,z) \geqslant C\delta^{-n-1}(z), \tag{3.5}$$

for any $z \in \Omega$. Moreover, for any $p > 1$, there is a constant $C(p,\Omega)$ such that

$$\|K(z,\cdot)\|_{L^p(\Omega)} \leqslant C\delta^{-(n+1)\left(1-\frac{1}{p}\right)}(z), \tag{3.6}$$

for any $z \in \Omega$, see e.g. [1, Theorem 2.7], also [26, 29].

Now, using properties of the Bergman projection, we have

$$\begin{aligned} \int_\Omega |K(z,w)|^2 \delta^\alpha (w) dw &= \int_\Omega K(w,z) \delta^\alpha (w) K(z,w) dw \\ &= \int_\Omega K(w,z) \delta^\alpha (w) \left(\int_\Omega K(z,\xi) K(\xi,w) d\xi \right) dw \\ &= \int_\Omega \left(\int_\Omega K(\xi,w) \delta^\alpha (w) K(w,z) dw \right) K(z,\xi) d\xi. \end{aligned}$$

By Hölder's inequality, it follows

$$\begin{aligned} \int_\Omega |K(z,w)|^2 \delta^\alpha (w) dw &\leqslant \left\| \int_\Omega K(\cdot,w) \delta^\alpha (w) K(w,z) dw \right\|_{L^q(\Omega)} \|K(z,\cdot)\|_{L^{q'}(\Omega)} \\ &\lesssim \|K(z,\cdot)\|_{L^p(\Omega)} \|K(z,\cdot)\|_{L^{q'}(\Omega)} \\ &\lesssim \delta^{-(n+1)\left(1-\frac{1}{p}+\frac{1}{q}\right)}(z). \end{aligned} \tag{3.7}$$

Here, the second inequality comes from the boundedness of T_α, the third follows from (3.6), and q' is the dual exponent of q, i.e. $1/q + 1/q' = 1$. On the other hand, by using Proposition 3.1, Proposition 3.2 and (3.5), we obtain

$$\int_\Omega |K(z,w)|^2 \delta^\alpha (w) dw \geqslant \int_{\{G(\cdot,z)<-1\}} |K(z,w)|^2 \delta^\alpha (w) dw$$

$$
\begin{aligned}
&\gtrsim \delta^{\alpha}(z)\left|\log\delta(z)\right|^{-\alpha}\int\limits_{\{G(\cdot,z)<-1\}}\left|K(z,w)\right|^{2}dw\\
&\gtrsim \delta^{\alpha}(z)\left|\log\delta(z)\right|^{-\alpha}K(z,z)\\
&\gtrsim \delta^{\alpha-n-1}(z)\left|\log\delta(z)\right|^{-\alpha}.
\end{aligned}
$$

From this and (3.7), the conclusion follows by letting $z\to\partial\Omega$. $\square$

Remark 3.5. A similar approach has been used in [23] for Hartogs triangle domains.

4. Proof of Theorem 1.1

Let us first recall some facts from the theory of Hardy spaces. We refer readers to the books by Stein [28] and Krantz [25] for details.

Let $\Omega\subset\mathbb{C}^n$ be a bounded domain with C^2 boundary. The Hardy space $h^2(\Omega)$ is defined by

$$
h^{2}(\Omega):=\left\{f \text{ harmonic on } \Omega:\int\limits_{\partial\Omega}\left|f(z)\right|^{2}d\sigma(z):=\limsup_{\varepsilon\to0^{+}}\int\limits_{\delta=\varepsilon}\left|f(z)\right|^{2}d\sigma(z)<\infty\right\}.
$$

There always exists a positive constant ε_0 depending on Ω such that the following norms are equivalent

$$
\left(\int\limits_{\partial\Omega}\left|f(z)\right|^{2}d\sigma(z)\right)^{\frac{1}{2}} \text{ and } \left(\sup_{0<\varepsilon<\varepsilon_0}\int\limits_{\delta=\varepsilon}\left|f(z)\right|^{2}d\sigma(z)\right)^{\frac{1}{2}},
$$

for $f\in h^2(\Omega)$. Moreover, there is a constant C depending on Ω such that

$$
\int\limits_{\Omega}\left|f(z)\right|^{2}dV(z)\leqslant C\int\limits_{\partial\Omega}\left|f(z)\right|^{2}d\sigma(z), \tag{4.1}
$$

for any $f\in h^2(\Omega)$. That is, the L^2-norm is dominated by the Hardy space norm for functions in $h^2(\Omega)$. Finally, we will need the following result, see [15, Lemma 2.2].

Lemma 4.1. *Let $\Omega\subset\mathbb{C}^n$ be a bounded domain with C^2 boundary. For any harmonic function u on Ω,*

$$
\limsup_{\varepsilon\to0^{+}}\int\limits_{\delta=\varepsilon}\left|u(z)\right|^{2}d\sigma(z)=\limsup_{r\to1^{-}}(1-r)\int\limits_{\Omega}\left|u(z)\right|^{2}\delta^{-r}(z)\,dV(z). \tag{4.2}
$$

We are ready to proceed Theorem 1.1.

Proof of Theorem 1.1. Let ε_0 and c_1 be positive constants such that

$$
\sup_{0<\varepsilon<\varepsilon_0}\int\limits_{\delta=\varepsilon}\left|f(z)\right|^{2}d\sigma(z)\leqslant c_1\int\limits_{\partial\Omega}\left|f(z)\right|^{2}d\sigma(z),\ \forall f\in h^2(\Omega).
$$

We first assume that $c_0\delta(w)<\varepsilon_0$, with $c_0:=(e+1)/(e-1)$. By applying (3.4) in Proposition 3.2,

$$
\begin{aligned}
\int\limits_{\{G(\cdot,w)<-1\}}\left|K(z,w)\right|^{2}dV(z) &\leqslant \int\limits_{\{\delta(\cdot)\leqslant c_0\delta(w)\}}\left|K(z,w)\right|^{2}dV(z)\\
&\leqslant \int\limits_{0}^{c_0\delta(w)}\left(\int\limits_{\delta=\varepsilon}\left|K(z,w)\right|^{2}d\sigma(z)\right)d\varepsilon \qquad (4.3)
\end{aligned}
$$

$$\leqslant \quad c_0 c_1 \delta(w) \int\limits_{\partial\Omega} |K(z,w)|^2 \, d\sigma(z).$$

By using (3.1) in Proposition 3.1,

$$\int\limits_{\{G(\cdot,w)<-1\}} |K(z,w)|^2 \, dV(z) \geqslant e^{-2n} K(w,w). \tag{4.4}$$

Combining (4.3) with (4.4), we conclude that

$$\|K(\cdot,w)\|_{L^2(\partial\Omega)} \geqslant C_1 \sqrt{\frac{K(w,w)}{\delta(w)}},$$

for a positive constant C_1 depending on Ω.

For the case $c_0\delta(w) \geqslant \varepsilon_0$, using (4.1) we have

$$\begin{aligned}
\int\limits_{\partial\Omega} |K(z,w)|^2 \, d\sigma(z) &\geqslant C \int\limits_{\Omega} |K(z,w)|^2 \, dV(z) \\
&= CK(w,w) \\
&\geqslant \frac{C\varepsilon_0}{c_0} \frac{K(w,w)}{\delta(w)}.
\end{aligned}$$

Therefore we have proved the left-hand side of (1.1).

We now turn to the proof of the right-hand side. Since Ω is convex, the function $-\delta$ is convex on Ω, and is thus also plurisubharmonic on Ω, see e.g. [2]. By applying Proposition 2.1 with $\psi(z) := -r\log(\delta(z))$, we have

$$(1-r) \int\limits_{\Omega} |P(f)(z)|^2 \delta^{-r}(z) \, dV(z) \leqslant \int\limits_{\Omega} |f(z)|^2 \delta^{-r}(z) \, dV(z), \tag{4.5}$$

for any $0 < r < 1$ and any measurable function f. Inserting

$$f(z) := \chi_{\{G(\cdot,w)<-t\}}(z) \, K_{\{G(\cdot,w)<-t\}}(z,w)$$

into (4.5), we obtain

$$(1-r) \int\limits_{\Omega} |K(z,w)|^2 \delta^{-r}(z) \, dV(z) \leqslant \int\limits_{\{G(\cdot,w)<-t\}} \left|K_{\{G(\cdot,w)<-t\}}(z,w)\right|^2 \delta^{-r}(z) \, dV(z), \tag{4.6}$$

for any $t > 0$. By Lemma 4.1, Proposition 3.1 and Proposition 3.2, it continues

$$\begin{aligned}
\int\limits_{\partial\Omega} |K(z,w)|^2 \, d\sigma(z) &= \limsup_{r\to 1^-} (1-r) \int\limits_{\Omega} |K(z,w)|^2 \delta^{-r}(z) \, dV(z) \\
&\leqslant \int\limits_{\{G(\cdot,w)<-t\}} \left|K_{\{G(\cdot,w)<-t\}}(z,w)\right|^2 \delta^{-1}(z) \, dV(z) \\
&\leqslant \frac{e^t+1}{e^t-1} \delta^{-1}(w) \int\limits_{\{G(\cdot,w)<-t\}} \left|K_{\{G(\cdot,w)<-t\}}(z,w)\right|^2 dV(z) \\
&= \frac{e^t+1}{e^t-1} \delta^{-1}(w) \, K_{\{G(\cdot,w)<-t\}}(w,w) \\
&\leqslant \frac{e^t+1}{e^t-1} e^{2nt} \frac{K(w,w)}{\delta(w)}.
\end{aligned}$$

The desired inequality then follows by noting that

$$\inf\left\{\left(e^t+1\right)e^{2nt}/\left(e^t-1\right): t>0\right\} < 4en+1.$$

$\square$

Remark 4.2. Since the constant $C_2=\sqrt{4en+1}$ depends only on the dimension n, it suggests for example a study of the sharp estimates in Theorem 1.1.

Remark 4.3. The method used in the proof of Theorem 1.1 can be extended to domains having a plurisubharmonic defining function, such as strongly pseudoconvex domains and Kohn special domains defined by

$$\Omega_F := \left\{z\in\mathbb{C}^n : |f_1(z)|^2+\ldots+|f_m(z)|^2<1\right\},$$

where $F=(f_1,\ldots,f_m):\mathbb{C}^n\to\mathbb{C}^m$ is a holomorphic map, see [12]. It is known that for strongly pseudoconvex domains and Kohn special domains, $\delta(z)\approx\delta(w)$ for $z\in\{G(\cdot,w)<-1\}$, see [17, 12, 15]. As a result, the estimate $\|K(\cdot,w)\|_{L^2(\partial\Omega)}\approx\left(K(w,w)/\delta(w)\right)^{1/2}$ holds true for these domains. For a general domain admitting a plurisubharmonic defining function, we may use the estimates in (3.3), which involve the logarithmic terms. To be precise, let us state these as the following corollary.

Corollary 4.4. *Let Ω be a bounded domain with C^2 boundary.*

(1) If Ω is either a strongly pseudoconvex domain or a Kohn special domain then there exist positive constants C_1 and C_2 such that for any $w\in\Omega$,

$$C_1\sqrt{\frac{K(w,w)}{\delta(w)}}\leqslant\|K(\cdot,w)\|_{L^2(\partial\Omega)}\leqslant C_2\sqrt{\frac{K(w,w)}{\delta(w)}}.$$

(2) If Ω is a pseudoconvex domain having a plurisubharmonic defining function then there exist positive constants C_1 and C_2 such that for any $w\in\Omega$,

$$C_1\sqrt{\frac{K(w,w)}{\delta(w)\left|\log\delta(w)\right|^n}}\leqslant\|K(\cdot,w)\|_{L^2(\partial\Omega)}\leqslant C_2\sqrt{\frac{K(w,w)\left|\log\delta(w)\right|}{\delta(w)}}.$$

References

[1] Abate, M., Raissy, J., and Saracco, A. (2012). Toeplitz operators and Carleson measures in strongly pseudoconvex domains. *J. Funct. Anal.*, 263(11):3449–3491.

[2] Armitage, D. H. and Kuran, U. (1985). The convexity of a domain and the superharmonicity of the signed distance function. *Proc. Amer. Math. Soc.*, 93(4):598–600.

[3] Bell, S. (1986). Differentiability of the Bergman kernel and pseudolocal estimates. *Math. Z.*, 192(3):467–472.

[4] Berndtsson, B. and Charpentier, P. (2000). A Sobolev mapping property of the Bergman kernel. *Math. Z.*, 235(1):1–10.

[5] Błocki, Z. (2004). A note on the Hörmander, Donnelly-Fefferman, and Berndtsson L^2-estimates for the $\overline{\partial}$-operator. *Ann. Polon. Math.*, 84(1):87–91.

[6] Błocki, Z. (2005). The Bergman metric and the pluricomplex Green function. *Trans. Amer. Math. Soc.*, 357(7):2613–2625.

[7] Błocki, Z. (2013). Suita conjecture and the Ohsawa-Takegoshi extension theorem. *Invent. Math.*, 193(1):149–158.

[8] Błocki, Z. (2015). Bergman kernel and pluripotential theory. In *Analysis, complex geometry, and mathematical physics: in honor of Duong H. Phong*, volume 644 of *Contemp. Math.*, pages 1–10. Amer. Math. Soc., Providence, RI.

[9] Boas, H. P. (1987). Extension of Kerzman's theorem on differentiability of the Bergman kernel function. *Indiana Univ. Math. J.*, 36(3):495–499.

[10] Boas, H. P. and Straube, E. J. (1991). Sobolev estimates for the $\overline{\partial}$-Neumann operator on domains in $\mathbf{C}^n$ admitting a defining function that is plurisubharmonic on the boundary. *Math. Z.*, 206(1):81–88.

[11] Catlin, D. W. Aimpl: Cauchy-Riemann equations in several variables, available at http://aimpl.org/crscv.

[12] Chen, B. and Fu, S. (2012). The reproducing kernels and the finite type conditions. *Illinois J. Math.*, 56(1):67–83 (2013).

[13] Chen, B.-Y. (2016). Parameter dependence of the Bergman kernels. *Adv. Math.*, 299:108–138.

[14] Chen, B.-Y. (2017). Bergman kernel and hyperconvexity index. *Anal. PDE*, 10(6):1429–1454.

[15] Chen, B.-Y. and Fu, S. (2011). Comparison of the Bergman and Szegö kernels. *Adv. Math.*, 228(4):2366–2384.

[16] Chen, S.-C. and Shaw, M.-C. (2001). *Partial differential equations in several complex variables*, volume 19 of *AMS/IP Studies in Advanced Mathematics*. American Mathematical Society, Providence, RI; International Press, Boston, MA.

[17] Diederich, K. and Herbort, G. (2000). Quantitative estimates for the Green function and an application to the Bergman metric. *Ann. Inst. Fourier (Grenoble)*, 50(4):1205–1228.

[18] Edholm, L. D. and McNeal, J. D. (2017). Bergman subspaces and subkernels: degenerate L^p mapping and zeroes. *J. Geom. Anal.*, 27(4):2658–2683.

[19] Grisvard, P. (2011). *Elliptic problems in nonsmooth domains*, volume 69 of *Classics in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA.

[20] Harrington, P. S. (2008). The order of plurisubharmonicity on pseudoconvex domains with Lipschitz boundaries. *Math. Res. Lett.*, 15(3):485–490.

[21] Herbort, G. (1999). The Bergman metric on hyperconvex domains. *Math. Z.*, 232(1):183–196.

[22] Khanh, T., Liu, J., and Thuc, P. (2017). Bergman-Toeplitz operators on weakly pseudoconvex domains. *arXiv:1710.10761*.

[23] Khanh, T., Liu, J., and Thuc, P. (2018). Bergman-Toeplitz operators on fat Hartogs triangles. *arXiv:1802.09174*.

[24] Kohn, J. J. (1963). Harmonic integrals on strongly pseudo-convex manifolds. I. *Ann. of Math. (2)*, 78:112–148.

[25] Krantz, S. G. (2001). *Function theory of several complex variables*. AMS Chelsea Publishing, Providence, RI. Reprint of the 1992 edition.

[26] Li, H. (1992). BMO, VMO and Hankel operators on the Bergman space of strongly pseudoconvex domains. *J. Funct. Anal.*, 106(2):375–408.

[27] Michel, J. and Shaw, M.-C. (2001). The $\overline{\partial}$-Neumann operator on Lipschitz pseudoconvex domains with plurisubharmonic defining functions. *Duke Math. J.*, 108(3):421–447.

[28] Stein, E. M. (1972). *Boundary behavior of holomorphic functions of several complex variables*. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo. Mathematical Notes, No. 11.

[29] Ž. Čučković, Z. and McNeal, J. D. (2006). Special Toeplitz operators on strongly pseudoconvex domains. *Rev. Mat. Iberoam.*, 22(3):851–866.

[30] Zwonek, W. (1999). On Bergman completeness of pseudoconvex Reinhardt domains. *Ann. Fac. Sci. Toulouse Math. (6)*, 8(3):537–552.

E-mail address: ttp754@uowmail.edu.au

Institute for Mathematics and its Applications, School of Mathematics and Applied Statistics, University of Wollongong, Wollongong, NSW 2522, AUSTRALIA.